CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems to usual applications.

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Collaboration at various stages of the work and in the framework of the Project *Evolution Equations in Combinatorics and Physics* :
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CIP seminar,
Friday conversations:
For this seminar, please have a look at Slide CCRT[n] & ff.

Outline

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

CCRT[21] MRS and the outer world II.

How infinite sums and product represent functions.

Let us first begin by a micro-compendium about the value of training to form the researcher (source [17], preface to the first edition).

It is said that Ramanujan taught himself mathematics by systematically working through 6000 problems[a] *of Carr's Synopsis of Elementary Results in Pure and Applied Mathematics.*
Freeman Dyson in his DISTURBING THE UNIVERSE *describes the mathematical days of his youth when he spent his summer months working through hundreds of problems in differential equations.*
If we look back at our own mathematical development, we can certify that problem solving plays an important role in the training of the research mind.
In fact, it would not be an exaggeration to say that the ability to do research is essentially the art of asking the "right" questions. I suppose Pólya summarized this in his famous dictum: "IF YOU CAN'T SOLVE A PROBLEM, THEN THERE IS AN EASIER PROBLEM YOU CAN SOLVE - FIND IT!"

[a] Actually, Carr's Synopsis is not a problem book. It is a collection of theorems used by students to prepare themselves for the Cambridge Tripos. Ramanujan made it famous by using it as a problem book.

TSC and metric (abelian) groups

1. Let us restart from Exercise 4 of CCRT[20].
   1. **Ex4.** –
      Let $(G,+,d)$ be an abelian group endowed with a distance d. We say that it is a metric group if the operations $(g,h)\to g+h$ and $g\to -g$ are continuous.
      1) Let $\mathcal{X}$ be an alphabet and $\mathbf{k}$ a ring. Prove that $(\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle,+,d)$, where d is the distance (**??**) is a metric group.
      2) In a metric group, a family $(g_i)_{i\in I}$ is said *summable*[a] to S if
      $$(\forall\epsilon>0)(\exists F\subset_{finite} I)(F\subset F'\subset_{finite} I\Longrightarrow d\Big(\sum_{j\in F'} g_j,S\Big)<\epsilon)$$
      3) Show that, if $\mathcal{X}$ is finite, a family $(S_i)_{i\in I}$ of series is summable if, for all $w\in\mathcal{X}^*$, the map $i\to\langle S_i|w\rangle$ is finitely supported. Show that its sum is then
      $$S=\sum_{w\in\mathcal{X}^*}\sum_{i\in I}\langle S_i|w\rangle\, w$$

[a] For summability, have a look there
https://mathoverflow.net/questions/289760
http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf

As a motivation: the mechanism of MRS (double series and linear operators)

1. And (re)consider the MRS factorization which is one of our precious jewels.

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \tag{1}$$

2. It is of the form $A = B = C = D$. What do we have ?
   - $A = B$ is a definition.
   - $B = C$ is the expression of "Bases in Duality" and is better interpreted as an **identity between operators**.
   - $C = D$ is a factorization into an infinite product again better interpreted as an identity between operators.
3. To understand (and prove) (1) the ultrametric distance (indeed a $\mathfrak{M}$-adic distance) will be sufficient. But first, let's have a slide of motivation.

Double series and operators

4. Let us start with a double series $S \in \mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$. It is expressed as

$$S = \sum_{u,v \in \mathcal{X}^*} \langle S | u \otimes v \rangle \, u \otimes v \tag{2}$$

5. This sum can be rearranged as

$$S = \sum_{v \in \mathcal{X}^*} \Big(\sum_{u \in \mathcal{X}^*} \langle S | u \otimes v \rangle \, u \Big) \otimes v = \sum_{v \in \mathcal{X}^*} \Big(\sum_{u \in \mathcal{X}^*} \langle S | u \otimes v \rangle \, u \Big) \underbrace{\bigotimes}_{\text{mind this step}} v \tag{3}$$

On the left of the big tensor, there are series and, on the right there are polynomials (monomials there), so we must clarify something here.

6. Although the arrow $\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \otimes \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$ i.e. (`tensor product of series towards double series`) is not into in general (see CIP 09/02/21) its restriction to $\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \otimes \mathbf{k}\langle \mathcal{X} \rangle$ is into (exercise LTTR, see below).

$B = C$ as an identity between operators/1

7. Let us unpack this ... and be careful
8. **Ex5.** –
   Let $T = \sum_{u,v \in \mathcal{X}^*} \langle T | u \otimes v \rangle\, u \otimes v \in \mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$ be double series.
   1) Show that, for all fixed $v \in \mathcal{X}^*$, the family $(\langle T | u \otimes v \rangle\, u)_{u \in \mathcal{X}^*}$ is summable in $\mathbf{k}\langle\langle \mathcal{X}^* \rangle\rangle$, let T_v denote its sum.
   2) Show that the following composition $\mathtt{Im} = \mathrm{nat} \circ (\mathrm{Id} \otimes \mathtt{j})$ is into

$$\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \otimes \mathbf{k}\langle \mathcal{X} \rangle \xrightarrow{\mathrm{Id} \otimes \mathtt{j}} \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \otimes \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \xrightarrow{\mathtt{nat}} \mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle \qquad (4)$$

   3) Show that $\Big(\mathtt{Im}(T_v \otimes v)\Big)_{v \in \mathcal{X}^*}$ is summable in $\mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$ and that its sum is precisely T.
   4) Adapt the preceding replacing $(v)_{v \in \mathcal{X}^*}$ by a basis $(Q_i)_{i \in I}$ of $\mathbf{k}\langle X \rangle$. In particular prove that T can be written uniquely

$$T = \sum_{i \in I} \mathtt{Im}(L_i \otimes Q_i) \qquad (5)$$

9. **Ex6.** –
Firstly, we consider a family $\mathcal{F} = (L_i \otimes Q_i)_{i \in I}$ as in (5). For now, we only suppose that $(Q_i)_{i \in I}$ is summable.
1) Prove that, for all $w \in \mathcal{X}^*$, the family $(\langle L_i | w \rangle Q_i)_{i \in I}$ is summable in $\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle$.
2) To such a family we associate $\Phi_{\mathcal{F}} \in \mathrm{Hom}(\mathbf{k}\langle \mathcal{X} \rangle, \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle)$, defined by

$$w \mapsto \sum_{i \in I} \langle L_i | w \rangle Q_i = \Phi_{\mathcal{F}}(w) \tag{6}$$

Prove that $\Phi_{\mathcal{F}}$ depends only on S, we denote it by Φ_S.
3) Show that correspondence $\mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle \to \mathrm{Hom}(\mathbf{k}\langle \mathcal{X} \rangle, \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle)$ is into.
4) If $(Q_i)_{i \in I}$ is a basis of $\mathbf{k}\langle \mathcal{X} \rangle$ and $(L_i)_{i \in I}$ its family of coordinates forms (defined by $\langle L_i | Q_j \rangle = \delta_{ij}$, we set $\mathcal{F}_1 = (L_i \otimes Q_i)_{i \in I}$, show that

$$\Phi_{\mathcal{F}_1} = j : \ \mathbf{k}\langle \mathcal{X} \rangle \hookrightarrow \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \tag{7}$$

$C = D$ product and then infinite product/1

10. Of course, one can bluntly rearrange terms of C to get D **we warn the reader that this is NOT a proof** because we do not have established commutative convergence (we have signs and remember [29]) nor a correct mechanism for convergence of the infinite product.
11. Let's go aside the classical construction of P_w by the Lyndon basis and S_w by the magic recursion that we recall now.

Lyndon basis and its dual

$$
\begin{array}{rcl}
P_x = & x & \text{for } x \in X, \\
P_\ell = & [P_s, P_r] & \text{for } \ell \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X} \text{ and } \sigma(\ell) = (s, r), \\
P_w = & P_{\ell_1}^{i_1} \ldots P_{\ell_k}^{i_k} & \text{for } w = \ell_1^{i_1} \ldots \ell_k^{i_k}, \ell_1 \succ \ldots \succ \ell_k,\ (\ell_i \in \mathcal{L}yn\mathcal{X}).
\end{array}
$$

$$
\begin{array}{rcl}
S_x = & x & \text{for } x \in \mathcal{X}, \\
S_l = & xS_u, & \text{for } l = xu \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X}, \\
S_w = & \dfrac{S_{l_1}^{\sqcup\sqcup\ i_1} \sqcup\sqcup \ldots \sqcup\sqcup S_{l_k}^{\sqcup\sqcup\ i_k}}{i_1! \ldots i_k!} & \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 \succ \ldots \succ l_k.
\end{array}
$$

$C = D$ product and then infinite product/2

12. In this particular case the route is

$$\texttt{Basis of Lie polynomials} \longrightarrow \texttt{PBW basis of NC polynomials} \longrightarrow \texttt{Recursion and dual basis.}$$

13. If we want to gain generality, we have to go first to φ-deformed shuffle products (see below the vast variety of such products present in the literature.

14. There is a common pattern.

$$\begin{aligned} & w \sqcup\!\sqcup_\varphi 1_{X^*} = 1_{X^*} \sqcup\!\sqcup_\varphi w = w \text{ and} \\ & au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a,b)\,(u \sqcup\!\sqcup_\varphi v) \end{aligned} \tag{8}$$

Variety of shuffles as found in literature

Name	Formula (recursion)	φ	Reference
Shuffle	$au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v)$	$\varphi \equiv 0$	Ree
Stuffle	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_i u \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$	Hoffman
Min-stuffle	$x_i u \sqcup\!\!\!- x_j v = x_i(u \sqcup\!\!\!- x_j v) + x_j(x_i u \sqcup\!\!\!- v) - x_{i+j}(u \sqcup\!\!\!- v)$	$\varphi(x_i, x_j) = -x_{i+j}$	Costermans
Muffle	$x_i u \sqcup\!\!\!\bullet\, x_j v = x_i(u \sqcup\!\!\!\bullet\, x_j v) + x_j(x_i u \sqcup\!\!\!\bullet\, v) + x_{i \times j}(u \sqcup\!\!\!\bullet\, v)$	$\varphi(x_i, x_j) = x_{i \times j}$	Enjalbert,HNM
q-shuffle	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	Bui
q-shuffle$_2$	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q^{i.j} x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	Bui
LDIAG$(1, q_s)$	$au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v) + q_s^{\lvert a \rvert \lvert b \rvert} a.b(u \sqcup\!\sqcup\, v)$	$\varphi(a, b) = q_s^{\lvert a \rvert \lvert b \rvert}(a.b)$	GD,Koshevoy,Penson,Tollu
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$	Chen-Fox-Lyndon
AC-stuffle	$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	Enjalbert,HNM
Semigroup--stuffle	$x_t u \sqcup\!\sqcup_\perp x_s v = x_t(u \sqcup\!\sqcup_\perp x_s v) + x_s(x_t u \sqcup\!\sqcup_\perp v) + x_{t \perp s}(u \sqcup\!\sqcup_\perp v)$	$\varphi(x_t, x_s) = x_{t \perp s}$	Deneufchâtel
φ-shuffle	$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$	$\varphi(a, b)$ law of AAU	Manchon, Paycha

$C = D$ product and then infinite product/2

15. In all tractable cases (stuffle and q-stuffle with constant or bicharacter or, even, cocycle).
    1. φ is commutative
    2. φ is dualizable and moderate
16. This means that the bialgebra $\left(\mathbf{k}\langle\mathcal{X}\rangle, \sqcup\!\sqcup_{\varphi}, 1_{\mathcal{X}^*}, \mathrm{conc}, \epsilon\right)$ is an enveloping algebra.
17. As we are in CCRT series, let us recall what is a universal enveloping algebra[a] in terms of categories and functors.

[a] i.e. is the most general (unital, associative) algebra that contains all representations of a Lie algebra, see [32].

Recall: CCRT[1,3] Universal Problems, heteromorphisms and adjunctions

Free structures w.r.t. a functor

1. Let $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ be two categories and $F : \mathcal{C}_{right} \to \mathcal{C}_{left}$ a (covariant) functor between them

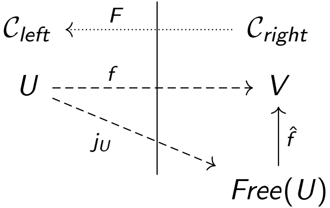
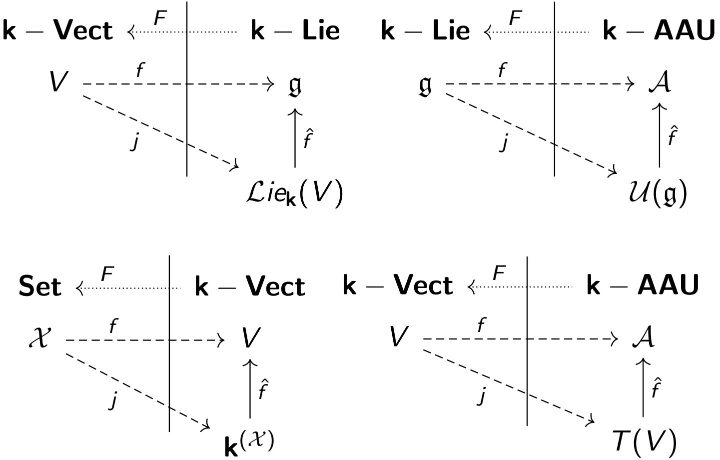


Figure: A solution of the universal problem w.r.t. the functor F is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$) such that, for all $f \in Hom(U, F[V])$ it exists a unique $\hat{f} \in Hom(Free(U), V)$ with $F[\hat{f}] \circ j_U = f$. Elements in $Hom(U, F[V])$ are called heteromorphisms their set is noted $Het_F(U, V)$.

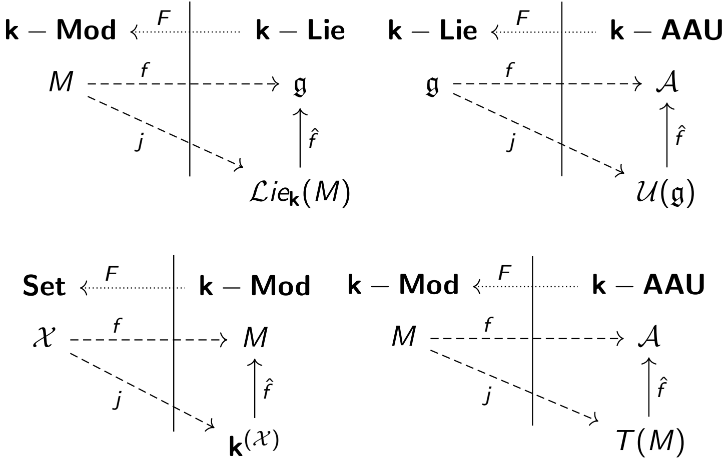
$$\big(\forall f \in Hom(U, F[V])\big)\big(\exists! \, \hat{f} \in Hom(Free(U), V)\big)\big(F(\hat{f}) \circ j_U = f\big)$$

First example: $T = UL$, $\mathbf{k}$ field-based.



$$T(V) = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}(V)) \qquad \mathbf{k}\langle\mathcal{X}\rangle = T(\mathbf{k}^{(\mathcal{X})})$$

First example: $T = UL$, ring-based.



$$T(M) = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}(M)) \qquad \mathbf{k}\langle\mathcal{X}\rangle = T(\mathbf{k}^{(\mathcal{X})})$$

Independence of characters w.r.t. polynomials.

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Independence of characters with respect to polynomials

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6

2

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors,
$\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U} \to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = \ker(\epsilon)$. We build the following filtrations $(N \geq 0)$

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N \geq -1$

Independence of characters w.r.t. polynomials./2

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra. We note ϵ its counit and set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations ($N \geq 0$)

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N \geq -1$

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing (in particular $\mathcal{U}_{-1}^* = \{0\}$, $\mathcal{U}_0^* = k.\epsilon$).
One shows easily that, for $p, q \geq 0$ (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 0} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Independence of characters w.r.t. polynomials./3

Now, we can state the

Theorem (From MO, k ring without zero divisors)

The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}^*_\infty$.

Remark

i) $\mathcal{U}^*_\infty$ is a commutative k-algebra.
ii) The title ("Independence of characters ...") comes from the fact that, with $(k\langle X\rangle, conc, 1)$ (non commutative polynomials), k a $\mathbb{Q}$-algebra (without zero divisors) and one of the usual comultiplications (with Δ_+ cocommutative and nilpotent, as co-shufflle, co-stuffle or - commutatively - deformed), if one takes $\mathfrak{g}$ as the space of primitive elements, we have $\mathcal{U}^* = k\langle\langle X\rangle\rangle$ (series) and $\mathcal{U}^*_\infty = k\langle X\rangle$.

Variety of shuffles as found in literature

Name	Formula (recursion)	φ	Reference
Shuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)$	$\varphi \equiv 0$	Ree
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Min-stuffle	$x_i u \,\underline{\sqcup}\, x_j v = x_i(u \,\underline{\sqcup}\, x_j v) + x_j(x_i u \,\underline{\sqcup}\, v) - x_{i+j}(u \,\underline{\sqcup}\, v)$	$\varphi(x_i, x_j) = -x_{i+j}$	Costermans
Muffle	$x_i u \,\boxdot\, x_j v = x_i(u \,\boxdot\, x_j v) + x_j(x_i u \,\boxdot\, v) + x_{i\times j}(u \,\boxdot\, v)$	$\varphi(x_i, x_j) = x_{i\times j}$	Enjalbert,HNM
q-shuffle	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	Bui
q-shuffle$_2$	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q^{i.j} x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	Bui
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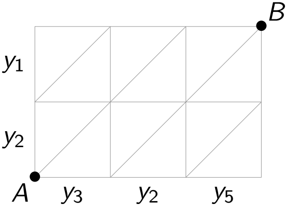
Common pattern

$$
\begin{aligned}
w \sqcup\!\sqcup_\varphi 1_{X^*} &= 1_{X^*} \sqcup\!\sqcup_\varphi w = w \text{ and} \\
au \sqcup\!\sqcup_\varphi bv &= a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)
\end{aligned} \tag{9}
$$

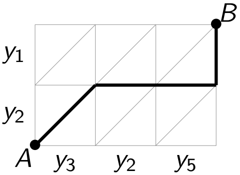
φ-shuffles as evaluations of paths

With $Y = \{y_i\}_{i\geq 1}$, one can see the product $u \sqcup\!\sqcup_\varphi v$ as a sum indexed by paths (with right-up-diagonal steps) within the grid formed by the two words (u horizontal and v vertical, the diagonal steps corresponding to the factors $\varphi(a,b)$)



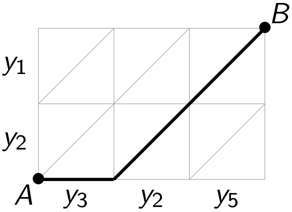
Computation of $y_2y_1 \sqcup\!\sqcup_\varphi y_3y_2y_5$

For example, the path



evaluates as $\varphi(y_2, y_3)y_2y_5y_1$

the path



reads $y_3 \varphi(y_2, y_2) \varphi(y_1, y_5)$.

We have the following

Theorem (Radford theorem for $ш_\varphi$)

Let **k** *be a* $\mathbb{Q}$*-algebra (associative, commutative with unit) such that*

$$ш_\varphi : \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle$$

is associative and commutative then

- $(\mathcal{L}yn(X)^{ш_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}}$ *is a linear basis of* $\mathbf{k}\langle X\rangle$.
- *This entails that* $(\mathbf{k}\langle X\rangle, ш_\varphi, 1_{X^*})$ *is a polynomial algebra with* $\mathcal{L}yn(X)$ *as transcendence basis.*

Making (combinatorial) bialgebras

Proposition

Let $\mathbf{k}$ *be a commutative ring (with unit). We suppose that the product* φ *is associative, then, on the algebra* $(\mathbf{k}\langle X\rangle, ⧢_{\varphi}, 1_{X^*})$*, we consider the comultiplication* Δ_{conc} *dual to the concatenation*

$$\Delta_{conc}(w) = \sum_{uv=w} u \otimes v \tag{10}$$

and the "constant term" character $\varepsilon(P) = \langle P | 1_{X^*}\rangle$.
Then

(i) *With this setting, we have a bialgebra* [a].

$$\mathcal{B}_{\varphi} = (\mathbf{k}\langle X\rangle, ⧢_{\varphi}, 1_{X^*}, \Delta_{conc}, \varepsilon) \tag{11}$$

(ii) *The bialgebra (eq. 11) is, in fact, a Hopf Algebra.*

[a] Commutative and, when $|X| \geq 2$, noncocommutative.

Dualizability

If one considers φ as defined by its structure constants

$$\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^{z}\,z$$

one sees at once that $\sqcup\!\sqcup_\varphi$ is dualizable within $\mathbf{k}\langle X\rangle$ iff the tensor $\gamma_{x,y}^{z}$ is locally finite in its contravariant place "z" i.e.

$$(\forall z\in X)(\#\{(x,y)\in X^2|\gamma_{x,y}^{z}\neq 0\}<+\infty)\ .$$

Remark

Shuffle, stuffle and infiltration are dualizable. The comultiplication associated with the stuffle with negative indices is not.

Dualizability/2

In the case when $\shuffle_{\varphi}$ is dualizable, one has a comultiplication

$$\Delta_{\shuffle_{\varphi}} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$$

such that, for all $u, v, w \in X^*$

$$\langle u \shuffle_{\varphi} v | w\rangle = \langle u \otimes v | \Delta_{\shuffle_{\varphi}}(w)\rangle \tag{12}$$

Then, the following

$$\mathcal{B}_{\varphi}^{\vee} = (\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{\shuffle_{\varphi}}, \varepsilon) \tag{13}$$

is a bialgebra in duality with $\mathcal{B}_{\varphi}$ (not always a Hopf algebra although $\mathcal{B}$ was so, for example, see $\mathcal{B}$ with $\shuffle_{\varphi} = \uparrow_q$ i.e. the q-infiltration).

The interest of these bialgebras is that they provide a host of easy-to-within-compute bialgebras with easy-to-implement-and-compute set of characters through the following proposition.

Proposition (Conc-Bialgebras)

Let $\mathbf{k}$ *be a commutative ring,* X *a set and* $\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^{z}\,z$ *an associative and dualizable law on* $\mathbf{k}.X$. *Let* $\sqcup\!\sqcup_{\varphi}$ *and* $\Delta_{\sqcup\!\sqcup_{\varphi}}$ *be the associated product and co-product. Then:*
i) $\mathcal{B}=(\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{\sqcup\!\sqcup_{\varphi}}, \epsilon)$ *is a bialgebra which, in case* $\mathbb{Q}\hookrightarrow\mathbf{k}$, *is an enveloping algebra iff* φ *is commutative and* $\Delta^{+}_{\sqcup\!\sqcup_{\varphi}}$ *nilpotent.*
ii) In the general case $S\in\mathbf{k}\langle\langle X\rangle\rangle=\mathbf{k}\langle X\rangle^{\vee}$ *is a character for* $\mathcal{A}=(\mathbf{k}\langle X\rangle, conc, 1_{X^*})$ *(i.e. a conc-character) iff it is of the form*

$$S=(\sum_{x\in X}\alpha_x\,x)^*=\sum_{n\geq 0}(\sum_{x\in X}\alpha_x\,x)^n \text{ and, with this notation} \tag{14}$$

$$(\sum_{x\in X}\alpha_x\,x)^*\sqcup\!\sqcup_{\varphi}(\sum_{x\in X}\beta_y\,y)^*=\left(\sum_{z\in X}(\alpha_z+\beta_z)\,z+\sum_{x,y\in X}\alpha_x\beta_y\,\varphi(x,y)\right)^* \tag{15}$$

GD, Darij Grinberg and Hoang Ngoc Minh *Three variations on the linear independence of grouplikes in a coalgebra*, [arXiv:2009.10970]

GD, Quoc Huan Ngô and V. Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

Main result about independence of characters w.r.t.

Theorem (G.D., Darij Grinberg, H. N. Minh)

Let $\mathcal{B}$ *be a* $\mathbf{k}$*-bialgebra. As usual, let* $\Delta = \Delta_{\mathcal{B}}$ *and* $\epsilon = \epsilon_{\mathcal{B}}$ *be its comultiplication and its counit.*
Let $\mathcal{B}_+ = \ker(\epsilon)$*. For each* $N \geq 0$*, let* $\mathcal{B}_+^N = \underbrace{\mathcal{B}_+ \cdot \mathcal{B}_+ \cdot \cdots \cdot \mathcal{B}_+}_{N \text{ times}}$*, where* $\mathcal{B}_+^0 = \mathcal{B}$*. Note that* $\left(\mathcal{B}_+^0, \mathcal{B}_+^1, \mathcal{B}_+^2, \ldots\right)$ *is called the* *standard decreasing filtration* *of* $\mathcal{B}$*.*
For each $N \geq -1$*, we define a* $\mathbf{k}$*-submodule* $\mathcal{B}_N^\vee$ *of* $\mathcal{B}^\vee$ *by*

$$\mathcal{B}_N^\vee = (\mathcal{B}_+^{N+1})^\perp = \left\{ f \in \mathcal{B}^\vee \mid f\left(\mathcal{B}_+^{N+1}\right) = 0 \right\}. \tag{16}$$

Thus, $\left(\mathcal{B}_{-1}^\vee, \mathcal{B}_0^\vee, \mathcal{B}_1^\vee, \ldots\right)$ *is an increasing filtration of* $\mathcal{B}_\infty^\vee := \bigcup_{N\geq -1} \mathcal{B}_N^\vee$ *with* $\mathcal{B}_{-1}^\vee = 0$*.*

Theorem (DGM, cont'd)

Let also $\Xi(\mathcal{B})$ be the monoid (group, if $\mathcal{B}$ is a Hopf algebra) of characters of the algebra $(\mathcal{B}, \mu_{\mathcal{B}}, 1_{\mathcal{B}})$.
Then:

(a) *We have $\mathcal{B}_p^\vee \circledast \mathcal{B}_q^\vee \subseteq \mathcal{B}_{p+q}^\vee$ for any $p, q \geq -1$ (where we set $\mathcal{B}_{-2}^\vee = 0$). Hence, $\mathcal{B}_\infty^\vee$ is a subalgebra of the convolution algebra $\mathcal{B}^\vee$.*

(b) *Assume that $\mathbf{k}$ is an integral domain. Then, the set $\Xi(\mathcal{B})^\times$ of invertible characters (i.e., of invertible elements of the monoid $\Xi(\mathcal{B})$) is left $\mathcal{B}_\infty^\vee$-linearly independent.*

Remark

The standard decreasing filtration of $\mathcal{B}$ is weakly decreasing, it can be stationary after the first step. An example can be obtained by taking the universal enveloping bialgebra of any simple Lie algebra (or, more generally, of any perfect Lie algebra); it will satisfy $\bigcap_{n\geq 0} \mathcal{B}_+^n = \mathcal{B}_+$.

Corollary

We suppose that $\mathcal{B}$ is cocommutative, and $\mathbf{k}$ *is an integral domain. Let $(g_x)_{x\in X}$ be a family of elements of $\Xi(\mathcal{B})^\times$ (the set of invertible characters of $\mathcal{B}$), and let $\varphi_X : \mathbf{k}[X] \to (\mathcal{B}^\vee, \circledast, \epsilon)$ be the* $\mathbf{k}$*-algebra morphism that sends each $x \in X$ to g_x. In order for the family $(g_x)_{x\in X}$ (of elements of the commutative ring $(\mathcal{B}^\vee, \circledast, \epsilon)$) to be algebraically independent over the subring $(\mathcal{B}^\vee_\infty, \circledast, \epsilon)$, it is necessary and sufficient that the monomial map*

$$\begin{aligned} m : \mathbb{N}^{(X)} &\to (\mathcal{B}^\vee, \circledast, \epsilon), \\ \alpha &\mapsto \varphi_X(X^\alpha) = \prod_{x\in X} g_x^{\alpha_x} \end{aligned} \tag{17}$$

(where α_x means the x-th entry of α) be injective.

Examples

Let **k** be an integral domain, and let us consider the standard bialgebra $\mathcal{B} = (\mathbf{k}[x], \Delta, \epsilon)$ For every $c \in \mathbf{k}$, there exists only one character of $\mathbf{k}[x]$ sending x to c; we will denote this character by $(c.x)^* \in \mathbf{k}[[x]]$ (motivation about this notation is Kleene star). Thus, $\Xi(\mathcal{B}) = ((c.x)^* \mid c \in \mathbf{k})$. It is easy to check that $(c_1.x)^* \sqcup\!\sqcup (c_1.x)^* = \left((c_1 + c_2).x\right)^*$ for any $c_i \in \mathbf{k}$ $(*)$. Thus, any $c_1, c_2, \ldots, c_k \in \mathbf{k}$ and any $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ satisfy

$$((c_1.x)^*)^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup ((c_2.x)^*)^{\sqcup\!\sqcup\, \alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup ((c_k.x)^*)^{\sqcup\!\sqcup\, \alpha_k} = \left((\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k).x\right)^* . \quad (18)$$

From $(*)$ above, the monoid $\Xi(\mathcal{B})$ is isomorphic with the abelian group $(\mathbf{k}, +, 0)$; in particular, it is a group, so that $\Xi(\mathcal{B})^\times = \Xi(\mathcal{B})$.

Examples/2

Take $\mathbf{k} = \overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$) and $c_n = \sqrt{p_n} \in \mathbf{k}$, where p_n is the n-th prime number. What precedes shows that the family of series $\big((\sqrt{p_n}x)^*\big)_{n\geq 1}$ is algebraically independent over the polynomials (i.e., over $\overline{\mathbb{Q}}[x]$) within the commutative $\overline{\mathbb{Q}}$-algebra $\big(\overline{\mathbb{Q}}[[x]], ш, 1\big)$. This example can be double-checked using partial fractions decompositions as, in fact, $(\sqrt{p_n}x)^* = \dfrac{1}{1-\sqrt{p_n}x}$ (this time, the inverse is taken within the ordinary product in $\mathbf{k}[[x]]$) and

$$\Big(\frac{1}{1-\sqrt{p_n}x}\Big)^{ш\, n} = \frac{1}{1-n\sqrt{p_n}x}.$$

Examples/3

The preceding example can be generalized as follows: Let $\mathbf{k}$ still be an integral domain; let V be a $\mathbf{k}$-module, and let $\mathcal{B} = \big(T(V), \mathrm{conc}, 1_{T(V)}, \Delta_{\boxtimes}, \epsilon\big)$ be the standard tensor conc-bialgebra[a] For every linear form $\varphi \in V^{\vee}$, there is an unique character φ^* of $\big(T(V), \mathrm{conc}, 1_{T(V)}\big)$ such that all $u \in V$ satisfy

$$\langle \varphi^* | u \rangle = \langle \varphi | u \rangle. \tag{19}$$

Again, it is easy to check[b] that $(\varphi_1)^* \sqcup\!\sqcup (\varphi_2)^* = \big(\varphi_1 + \varphi_2\big)^*$ for any $\varphi_1, \varphi_2 \in V^{\vee}$, because both sides are characters of $\big(T(V), \mathrm{conc}, 1_{T(V)}\big)$ so that the equality has only to be checked on V.

---

[a] The one defined by

$$\Delta_{\boxtimes}(1) = 1 \otimes 1 \text{ and } \Delta_{\boxtimes}(u) = u \otimes 1 + 1 \otimes u \ ; \ \epsilon(u) = 0 \text{ for all } u \in V.$$

[b] For this bialgebra $\sqcup\!\sqcup$ stands for $\circledast$ on the space $\mathrm{Hom}(\mathcal{B}, \mathbf{k})$.

Examples/4

Again, from this, any $\varphi_1, \varphi_2, \ldots, \varphi_k \in V^\vee$ and any $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ satisfy

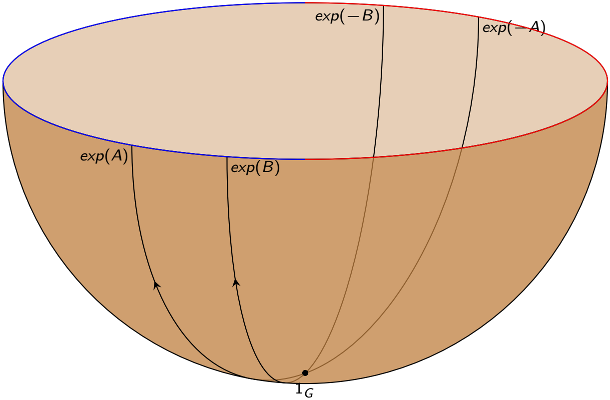
$$\left((\varphi_1)^*\right)^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup \left((\varphi_2)^*\right)^{\sqcup\!\sqcup\, \alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left((\varphi_k)^*\right)^{\sqcup\!\sqcup\, \alpha_k} = \left(\alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_k\varphi_k\right)^* . \quad (20)$$

The decreasing filtration of $\mathcal{B}$ is given by $\mathcal{B}_+^n = \bigoplus_{k\geq n} T_k(V)$ (the ideal of tensors of degree $\geq n$) and the reader may check easily that, in this case, $\mathcal{B}_\infty^\vee$ is the shuffle algebra of finitely supported linear forms i.e., for each $\Phi \in \mathcal{B}^\vee$, we have the equivalence

$$\Phi \in \mathcal{B}_\infty^\vee \iff (\exists N \in \mathbb{N})(\forall k \geq N)(\Phi(T_k(V)) = \{0\}).$$

Then, Corollary above shows that $(\varphi_i^*)_{i\in I}$ are $\mathcal{B}_\infty^\vee$-algebraically independent within $(T(V)^\vee, \sqcup\!\sqcup, \epsilon)$ iff the corresponding monomial map is injective, and (20) shows that it is so iff the family $(\varphi_i)_{i\in I}$ of linear forms is $\mathbb{Z}$-linearly independent in $V^\vee$.

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{\sqcup\sqcup}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

Hausdorff group of the stuffle Hopf algebra.

With $Y = \{y_i\}_{i \geq 1}$ and

$$\Delta_{\boxplus}(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j$$

the bialgebra $\mathcal{B} = (\mathbf{k}\langle X \rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\boxplus}, \epsilon)$ is an enveloping algebra (it is cocommutative, connex and graded by the weight function given by $||y_{i_1} y_{i_2} \cdots y_{i_k}|| = \sum_{s=1}^{k} i_s$ on a word $w = y_{i_1} y_{i_2} \cdots y_{i_k}$).
With $\varphi(y_i, y_j) = y_{i+j}$, (eq.15) gives

$$(\sum_{i \geq 1} \alpha_i \, y_i)^*_{\boxplus} (\sum_{j \geq 1} \beta_j \, y_j)^* = (\sum_{i \geq 1} \alpha_i \, y_i + \sum_{j \geq 1} \beta_j \, y_j + \sum_{i,j \geq 1} \alpha_i \beta_j \, y_{i+j})^* \qquad (21)$$

This formula suggests us to code, in an umbral style, $\sum_{k \geq 1} \alpha_k \, y_k$ by the series $\sum_{k \geq 1} \alpha_k \, x^k \in \mathbf{k}_+[[x]]$. Indeed, we get the following proposition whose first part, characteristic-freely describes the group of characters $\Xi(\mathcal{B})$ and its law and the second part, about the exp-log correspondence, requires $\mathbf{k}$ to be $\mathbb{Q}$-algebra.

Proposition

Let π_Y^{Umbra} be the linear isomorphism $\mathbf{k}_+[[x]] \to \widehat{\mathbf{k}.Y}$ defined by

$$\sum_{n\geq 1} \alpha_n\, x^n \mapsto \sum_{k\geq 1} \alpha_k\, y_k \tag{22}$$

Then

1. *One has, for $S, T \in \mathbf{k}_+[[x]]$,*

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \tag{23}$$

2. *From now on $\mathbf{k}$ is supposed to be a $\mathbb{Q}$-algebra.*
   For $t \in \mathbf{k}$ and $T \in \mathbf{k}_+[[x]]$, the family $(\frac{(t.T)^n}{n!})_{n\geq 0}$ is summable and one sets

$$G(t) = (\pi_Y^{Umbra}(e^{t.T} - 1))^* \tag{24}$$

Proposition (Cont'd)

3. *The parametric character G fulfills the "stuffle one-parameter group" property i.e. for $t_1, t_2 \in \mathbf{k}$, we have*

$$G(t_1 + t_2) = G(t_1) \uplus G(t_2) \; ; \; G(0) = 1_{Y^*} \tag{25}$$

4. *We have*

$$G(t) = exp_{\uplus}\left(t.\pi_Y^{Umbra}(T)\right) \tag{26}$$

5. *In particular, calling π_x^{Umbra} the inverse of π_Y^{Umbra} we get, for $P^* \in \Xi(\mathcal{B})$ (in other words $P \in \widehat{\mathbf{k}.Y}$),*

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1 + \pi_x^{Umbra}(P))) \tag{27}$$

Proof (Sketch)

i) We have

$$\pi_Y^{Umbra}(S) = \sum_{i\geq 1}\langle S|x^i\rangle y_i \quad \pi_Y^{Umbra}(T) = \sum_{j\geq 1}\langle T|x^j\rangle y_j$$

and then

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\sum_{i\geq 1}\langle S|x^i\rangle y_i)^* \uplus (\sum_{j\geq 1}\langle T|x^j\rangle y_j) =$$
$$\Big(\sum_{i\geq 1}\langle S|x^i\rangle y_i) + \sum_{j\geq 1}\langle T|x^j\rangle y_j + \sum_{i,j\geq 1}\langle S|x^i\rangle\langle T|x^j\rangle y_{i+j}\Big)^* =$$
$$(\pi_Y^{Umbra}(S+T+ST))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^*$$

ii.1) The one parameter group property is a consequence of (23) applied to the series $e^{t_i.T}-1,\ i=1,2$.

Proof (Sketch)/2

ii.2) Property 25 holds for every $\mathbb{Q}$-algebra, in particular in $\mathbf{k}_1 = \mathbf{k}[t]$ and $\mathbf{k}_1\langle\langle Y \rangle\rangle$ is endowed with the structure of a differential ring by term-by-term derivations (see [**?**] for formal details). We can write $G(t) = 1 + t.G_1 + t^2.G_2(t)$ (where $G_1 = \pi_Y^{Umbra}(T)$ is independent from t) and from 25, we have

$$G'(t) = G_1.G(t)\ ;\ \ G(0) = 1_{Y^*} \tag{28}$$

but $H(t) = exp_{\uplus}(t.G_1)$ satisfies 28 whence the equality.
ii.3) At $t = 1$, we have $\exp_{\uplus}(\pi_Y^{Umbra}(T)) = (\pi_Y^{Umbra}(e^T - 1))^*$ hence, with $P = \pi_Y^{Umbra}(e^T - 1)$ (take $T := \log(\pi_x^{Umbra}(P) + 1)$)

$$\pi_Y^{Umbra}(T) = \log_{\uplus}(P^*) \qquad \text{[QED]} \tag{29}$$

Application of (27)

$$(ty_k)^* = \exp_{\uplus}\Big(\sum_{n\geq 1} \frac{(-1)^{n-1} t^n y_{nk}}{n}\Big) \tag{30}$$

Conclusion(s): More applications and perspectives.

We have seen

1. Star of the plane property (see [12]) holds for non-commutative valued (as matrix-valued) characters.
2. Combinatorial study of other $\sqcup\!\sqcup_{\varphi}$ one-parameter groups and evolution equations in convolution algebras.
3. Factorisation of $\mathcal{A}$-valued characters ($\mathcal{A}$ **k**-CAAU).
   For example, with

$$\mathcal{B} = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{conc}, \epsilon)\,,\ \mathcal{A} = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})\,,\ \chi = Id$$

(χ is a shuffle character) one has (MRS factorisation)

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w)\otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \tag{31}$$

Conclusion(s): More applications and perspectives./2

4. Deformed version of factorisation above for $\shuffle_\varphi$ (with φ associative, commutative, dualisable and moderate). With

$$\mathcal{B} = \left(\mathbf{k}\langle X\rangle, \shuffle_\varphi, 1_{X^*}, \Delta_{conc}, \epsilon\right), \ \mathcal{A} = \left(\mathbf{k}\langle X\rangle, \shuffle_\varphi, 1_{X^*}\right), \ \chi = Id$$

(χ is a shuffle character) one has

$$\Gamma(\chi) = \sum_{w \in X^*} Id(w) \otimes w = \sum_{w \in X^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \qquad (32)$$

5. Holds for all enveloping algebras which are free as **k**-modules (with $\mathbb{Q} \hookrightarrow \mathbf{k}$). This could help to the combinatorial study of the group of characters of enveloping algebras of Lie algebras like KZ[a]-Lie algebras and other ones, or deformed.

[a] Knizhnik–Zamolodchikov.

Conclusion(s)/3: Main theorem

Theorem, [13]

Let $\mathbf{k}$ be a $\mathbb{Q}$-algebra and $\mathfrak{g}$ be a Lie algebra which is free as a $\mathbf{k}$-module. Let us fix an ordered basis $B = (b_i)_{i\in I}$ (where the ground set $(I, <)$ is totally ordered) of $\mathfrak{g}$. To construct the associated PBW basis of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, we use the following multiindex notation. For every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^{\alpha} = b_{i_1}^{\alpha(i_1)} \cdots b_{i_n}^{\alpha(i_n)} \in \mathcal{U} \tag{33}$$

where $\{i_1, \cdots, i_n\} \supset \mathrm{supp}(\alpha)$ (and $i_1 < \cdots < i_n$).
Consider the linear coordinate forms $B_\beta \in \mathcal{U}^\vee$ defined by

$$\langle B_\beta | B^\alpha \rangle = \delta_{\alpha,\beta}. \tag{34}$$

We will also use the elementary multiindices $e_i \in \mathbb{N}^{(I)}$ defined for all $i \in I$ by $e_i(j) = \delta_{i,j}$.

Conclusion(s): Main theorem/2

Theorem cont'd

Then:[a]

1. We have
$$B_\alpha \circledast B_\beta = \frac{(\alpha+\beta)\,!}{\alpha\,!\,\beta\,!} B_{\alpha+\beta} \tag{35}$$
and
$$B_{\alpha(i_1)e_{i_1}+\cdots+\alpha(i_k)e_{i_k}} = \frac{B_{e_{i_1}}^{\circledast\alpha(i_1)} \circledast \cdots \circledast B_{e_{i_k}}^{\circledast\alpha(i_k)}}{\alpha(i_1)\,!\cdots\alpha(i_k)\,!}. \tag{36}$$

2. The following infinite product identity holds:
$$Id_{\mathcal{U}} = \circledast_{i\in I}^{\rightarrow}\; e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} = \overrightarrow{\prod_{i\in I}}\; e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} \tag{37}$$
within $End(\mathcal{U})$.

[a] We use the notation $\alpha!$ for $\alpha\in\mathbb{N}^{(I)}$; this is the product $\alpha! = \prod_{i\in I}\alpha_i!$.

THANK YOU FOR YOUR ATTENTION !

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